



5.1 The inner product for finite-dimensional vectors

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Introduction

- In this topic, we will
 - Define the inner product for finite-dimensional real vectors $\mathbf{R}^n$
 - Consider some of the properties of the inner product
 - Define an appropriate inner product for
finite-dimensional complex vectors $\mathbf{C}^n$
 - Define inner product spaces





Inner products in $\mathbf{R}^n$

- If $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$, we will define the inner product of $\mathbf{u}$ and $\mathbf{v}$ as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k v_k$$





Inner products in $\mathbf{R}^n$

- For example, if $\mathbf{u} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$, then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= 3 \cdot 1 + (-5) \cdot 4 + 2 \cdot 6 \\ &= 6 - 20 + 12 = -2\end{aligned}$$





Properties of inner products in $\mathbf{R}^n$

- What are the useful properties of the inner product?

- First,
$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= \sum_{k=1}^n u_k v_k \\ &= \sum_{k=1}^n v_k u_k = \langle \mathbf{v}, \mathbf{u} \rangle\end{aligned}$$

- Thus, the inner product is *symmetric*
 - Recall we described equality as symmetric:
$$x = y \text{ if and only if } y = x$$





Properties of inner products in $\mathbf{R}^n$

- What are the useful properties of the inner product?

- Second, $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n \alpha u_k v_k$

$$= \alpha \sum_{k=1}^n u_k v_k = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$$

$$\begin{aligned} \langle \mathbf{u}_1 + \mathbf{u}_2, \mathbf{v} \rangle &= \sum_{k=1}^n (u_{1,k} + u_{2,k}) v_k \\ &= \sum_{k=1}^n (u_{1,k} v_k + u_{2,k} v_k) \\ &= \sum_{k=1}^n u_{1,k} v_k + \sum_{k=1}^n u_{2,k} v_k = \langle \mathbf{u}_1, \mathbf{v} \rangle + \langle \mathbf{u}_2, \mathbf{v} \rangle \end{aligned}$$





Properties of inner products in $\mathbf{R}^n$

- What are the useful properties of the inner product?
 - This second property is usually summarized as a single expression:

$$\langle \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2, \mathbf{v} \rangle = \alpha_1 \langle \mathbf{u}_1, \mathbf{v} \rangle + \alpha_2 \langle \mathbf{u}_2, \mathbf{v} \rangle$$

- Also

$$\langle \mathbf{u}, \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 \rangle = \beta_1 \langle \mathbf{u}, \mathbf{v}_1 \rangle + \beta_2 \langle \mathbf{u}, \mathbf{v}_2 \rangle$$

- We say that the inner product is *bilinear*,
that is, linear in both operands





Properties of inner products in $\mathbf{R}^n$

- What are the useful properties of the inner product?

- Third, $\langle \mathbf{u}, \mathbf{u} \rangle = \sum_{k=1}^n u_k u_k$
 $= \sum_{k=1}^n u_k^2 \geq 0$

- Also, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

- Thus, we say the inner product is *positive definite*





Properties of inner products in $\mathbf{R}^n$

- Thus, the properties are:
 - Symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
 - Bilinearity: $\langle \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2, \mathbf{v} \rangle = \alpha_1 \langle \mathbf{u}_1, \mathbf{v} \rangle + \alpha_2 \langle \mathbf{u}_2, \mathbf{v} \rangle$
 $\langle \mathbf{u}, \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 \rangle = \beta_1 \langle \mathbf{u}, \mathbf{v}_1 \rangle + \beta_2 \langle \mathbf{u}, \mathbf{v}_2 \rangle$
 - Positive definiteness: $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$,
and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$





Inner products in $\mathbf{C}^n$

- If $\mathbf{u}, \mathbf{v} \in \mathbf{C}^n$, can we define the inner product of $\mathbf{u}$ and $\mathbf{v}$ as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k v_k ?$$

- Consider $\mathbf{u} = \begin{pmatrix} 1 \\ j \end{pmatrix}$, so $\langle \mathbf{u}, \mathbf{u} \rangle = 1 \cdot 1 + j \cdot j$
 $= 1 - 1 = 0$





Inner products in $\mathbf{C}^n$

- If $\mathbf{u}, \mathbf{v} \in \mathbf{C}^n$, can we define the inner product of $\mathbf{u}$ and $\mathbf{v}$ as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k^* v_k$$

- Thus, for any vector $\mathbf{u}$, we have

$$\begin{aligned} \langle \mathbf{u}, \mathbf{u} \rangle &= \sum_{k=1}^n u_k^* u_k \\ &= \sum_{k=1}^n |u_k|^2 \geq 0 \end{aligned}$$

- Note that this, again, equals zero if and only if $\mathbf{u} = \mathbf{0}$





Inner products in $\mathbf{C}^n$

- For example, if $\mathbf{u} = \begin{pmatrix} 1+4j \\ 2-3j \\ 5-2j \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3+4j \\ 1-2j \\ 6+5j \end{pmatrix}$, then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= (1+4j)^* (-3+4j) + (2-3j)^* (1-2j) + (5-2j)^* (6+5j) \\ &= (1-4j)(-3+4j) + (2+3j)(1-2j) + (5+2j)(6+5j) \\ &= (13+16j) + (8-j) + (20+37j) \\ &= 41+52j\end{aligned}$$





Properties of inner products in $\mathbf{C}^n$

- Thus, the properties are:
 - Conjugate symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$
 - Sesquilinearity: $\langle \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2, \mathbf{v} \rangle = \alpha_1^* \langle \mathbf{u}_1, \mathbf{v} \rangle + \alpha_2^* \langle \mathbf{u}_2, \mathbf{v} \rangle$
 $\langle \mathbf{u}, \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 \rangle = \beta_1 \langle \mathbf{u}, \mathbf{v}_1 \rangle + \beta_2 \langle \mathbf{u}, \mathbf{v}_2 \rangle$
 - Positive definiteness: $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$,
and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$





Inner product spaces

- A vector space with an inner product is often described as an *inner product space*
 - The assumption is made that it must be a vector space





Notation

- Physicists and engineers define the complex inner product as:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k^* v_k$$

- Mathematicians define it as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^n u_k v_k^*$$





Summary

- Following this topic, you now
 - Know the definition of the real inner product
 - Are aware of the properties of the real inner product
 - Know that for complex vectors, the first vector is conjugated
 - This leads to some subtle changes to the properties





References

- [1] [https://en.wikipedia.org/wiki/Norm_\(mathematics\)](https://en.wikipedia.org/wiki/Norm_(mathematics))





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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